The Enemy of My Enemy Is My Friend: A New Condition for Two-sided Matching with Complementarities *

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Abstract

Stable matchings in the presence of complementarities need not exist. With a canonical form of payoff functions, the conventional sufficient condition for existence substantially restricts the range of pairwise complementarity/substitutability values. This paper provides a new sufficient condition for existence on the *maximal range* of complementarities/substitutabilities. Our condition also allows for a new class of settings that are relevant to labor markets but violate the conventional condition. We further show that under a minor assumption, our condition is also necessary to *guarantee* existence. Note that the preferences in our model are not covered by Baldwin and Klemperer (forthcoming).

Keywords: Matching with Complementarities, Core

JEL Codes: C62, C68, C71, C78, D44, D47, D50

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1 Introduction

It is known that in the presence of general complementarities/substitutabilities, a welfare-maximizing allocation of the economy with indivisible goods need not have the core property[1], an indicator for the stability of the market. The stability of markets can be important since it keeps markets robust and supports their long-term sustainability (Roth (2002)). On the other hand, there are many markets, such as labor markets and supply chain networks in manufacturing, with complementarities/substitutabilities that involve indivisible goods.

The matching literature provides an understanding of how restrictions on preferences over complementarities/substitutabilities ensure the existence of a stable matching[2]. However, the literature does not discuss how such conditions would constrain numerical values of complementarities/substitutabilities[3]. With a canonical form of utility functions involving pairwise complementarities, this paper will (a) show that the conventional sufficient conditions for the existence of a stable matching substantially restrict the *range* of complementarities, (b) provide a new sufficient condition (for existence in

[1]See Shapley and Scarf (1974). A simple example in a cooperative game is as follows. Suppose a surplus of 2.9 will be generated when any pair of the three agents form a coalition, while a surplus of 3.0 will be generated when all three agents form a coalition. Notice that the grand coalition is welfare-maximizing but two of the three agents always form a blocking coalition.

[2]See, for example, Kelso and Crawford (1982), Roth (1984), Hatfield and Milgrom (2005), Bikhchandani and Mamer (1997), Gul et al. (1999), Gul et al. (2000), Milgrom (2000), Bikhchandani and Ostroy (2002), Bikhchandani et al. (2002), Ausubel et al. (2006), Sun and Yang (2006), Sun and Yang (2009), Ostrovsky (2008), Echenique and Oviedo (2006), Pycia (2012), Hatfield et al. (2013), Kojima et al. (2013), Azevedo et al. (2013), Sun and Yang (2014), Azevedo and Hatfield (2015), Che et al. (2019).

[3]Notable exceptions are Murota and Shioura (2004) and Murota et al. (2016), though these papers have domains that are often too large for economic questions in the matching literature. The excessively large domain would often lead to a requirement of stronger conditions that tend to be uninteresting to economists, as we demonstrate in Section 3.

a one-to-many matching market with transferable utility) that has no range restriction and that tolerates a new class of settings relevant to labor markets, and (c) establish that with a minor assumption, our condition is necessary and sufficient to *guarantee* that such a stable matching exists.

Our condition is two-fold. First, all firms agree on which pair of workers go well (poorly) together. Second, workers can be partitioned into groups such that within a group, workers are complements, and across groups, workers are substitutes.

To understand our condition more precisely, consider a labor market with three workers, A, B, and C, and two firms, 1 and 2. Both firms care about joint production produced by an individual firm-worker match. To denote joint production generated by, for example, worker A and firm 1, we write $\phi_{A,1}$. Also, both firms care about extra (positive or negative) production generated by a pair of workers: $\{A, B\}, \{A, C\}, \{B, C\}$. To denote extra production generated by, for instance, $\{A, B\}$ at firm 1, we write $q^1_{A,B}$.

Similarly, workers care about some firm-specific value that a worker can gain from an individual firm-worker match[4]. To denote value that for example, worker A gets from working at firm 1, we write $\nu_{A,1}$. Also, every worker is concerned with whom their coworkers are in a pairwise manner at a specific firm. To denote utility that for instance, both worker A and B enjoy through working together at firm 1, we write $d^1_{A,B}$.

We call the sum of values generated by an individual firm-worker match, *total individual surplus.* To denote total individual surplus from, for example, a match between worker A and firm 1, we write a^1_A that is equal to $\phi_{A,1} + \nu_{A,1}$. Meanwhile, we call the sum of values produced by joint employment of a pair of workers, *total pairwise surplus.* To denote total pairwise surplus that for

[4]One example is the value of the human capital a worker accumulates at a firm.

instance, firm 1 and $\{A, B\}$ pair generate together, we write b^1_{AB} that is equal to $q^1_{A,B} + d^1_{A,B}$.

Our first requirement is that if $b^k_{ij} > 0$, then $b^{k'}_{ij} \geq 0$. For example, if $b^1_{AB} > 0$, then $b^2_{AB} \geq 0$. This is the so-called *sign-consistency* condition introduced by Candogan et al. (2015). In words, all firms agree upon *which pair of workers go well (poorly) together*[5].

The second requirement is that the agents can be partitioned into groups such that within a group, agents are pairwise complements ($b^k_{ij} \geq 0$), and across groups, agents are pairwise substitutes ($b^k_{ij} < 0$). This is the so-called *sign-balance* condition[6]. The sign-balance condition is satisfied if every cycle in a graph of total pairwise surplus (where nodes are workers and edges are total pairwise surplus b^k_{ij}) has an even number of negative edges. The clusterability property of a sign-balance graph implies that workers can be divided into (multiple) groups, within which workers are complements and across which workers are substitutes. Colloquially, *the enemy of my enemy is my friend.*

Note that the sign-consistency condition is a more generalized version of the assumption from Sun and Yang (2006) that the pattern of divisions of goods is shared by all the agents in the economy. Also, note that in our model, the sign-consistency condition is less restrictive than the sign-consistency assumption in papers like Candogan et al. (2015). Our model allows a pair of workers (say, A and B) to be complements at firm 1 and substitutes at firm 2, as long

[5]Note that the sign-consistency condition is quite different from the alignment idea in Banerjee et al. (2001) and Pycia (2012). The alignment concept essentially renders multiple agents into a single decision maker, while the sign-consistency assumption does not need to convert multiple agents to such a single decision maker. For example, firm k regards the synergy between a pair of workers extremely high (say, $b^k_{ij} = 100$), while another firm, k' perceives no synergy (say, $b^k_{ij} = 0$). Then, it is likely that the pair of workers ranks pretty high within the firm k's ranking over all the possible pairs of workers in the economy, while that pair of workers probably ranks low for firm k'.

[6]In the engineering literature, it is sometimes called the *structurally balanced* condition.

as these workers become more productive by working together (e.g., a couple) than working alone at firm 2 to compensate the firm. Conversely, a pair of workers can like working together at firm 1 but dislike working together at firm 2, as long as together they generate enough joint production at firm 2 to make the total pairwise surplus nonnegative.

In addition to these conditions, the main source of our results is the restriction on every agent's preferences with transferable utility to be so-called *binary quadratic programs* (BQP) preferences. In BQP preferences, an individual agent's utility function is quasi-linear and has utility from an individual match and utility from a pairwise complementarities/substitutabilities in an *additively separable* manner, so that utilitarian total welfare can be expressed as $\sum_k \sum_i a_i^k + \sum_k \sum_{i \neq j} b_{ij}^k$. First, allowing utility to be transferable from one agent to another partially reconciles conflicts of interests between two groups of coalitions. While pairwise complementarity is a usual restriction in the matching and exchange economy literature with indivisible goods[7], the literature usually imposes no assumption on the functional form of utility other than quasi-linearity in numeraire goods. The restriction to BQP preferences allows us to explicitly express parameters on the simultaneous consumption of a pair of goods, and consequently allows us to find how to group workers in such a way to prevent a blocking coalition.

We argue that a BQP preference is a natural way to restrict preferences with pairwise complementarities. Ausubel et al. (1997), for example, uses a form of BQP preferences in spectrum auction settings with pairwise complementarities, while Bertsimas et al. (1999), Candogan et al. (2015), and Candogan et al. (2018) deal with combinatorial auction problems. Furthermore, some papers in both the theoretical and empirical strategic interaction

[7]The famous gross-substitutes condition is conventionally imposed on a pair of goods.

literature such as Jackson and Wolinsky (1996), Jackson and Rogers (2005), and de Paula et al. (2014) use a form (or extended form) of BQP preferences as discussed in Jackson (2010)[8].

Furthermore, we believe that BQP preferences are intuitive when complementarities/substitutabilities are pairwise, due to the structure of simple addition of each individual and pairwise term. If some multiplicative structure in utility function is allowed, just one tiny negativity can overturn all the positivities. Yet, it is hard to think that for example, a small negative personal tension between a pair of authors would cancel out the entire quality of individual research work or other colleagues' work at some university. Meanwhile, we do admit the limitation of BQP preferences. For instance, BQP preferences do not capture influence of one or more workers to other workers. Some workers' individual productivity can be affected by a negative mood created by severe tension among other workers in the office.

We briefly delineate the significance and relevance of our condition to two-sided matching settings such as labor markets. We do so by comparing our condition with the gross substitutes and complements (GSC) condition suggested by Sun and Yang (2006). The GSC condition allows for preference structures more general than BQP preferences and is satisfied if goods can be divided into two groups, and within groups, goods are gross substitutes, and across groups, goods are gross complements[9]. However, in general, two-sided matching settings, the condition would imply that a friend of my friend has to be my enemy, which seems less plausible than ours in settings that involve humans (including countries and institutions) on both sides. Whether in school,

[8]A quadratic valuation objective function is one of the most canonical forms used for network-related research in the engineering literature.

[9]Note that while the GSC condition may sound *qualitatively* opposite to our condition, precisely speaking, it is not mathematically opposite.

the workplace, a community, a country, or the globe, we tend to see that one joins a group of people with whom he or she finds comfortable, and across these groups, people fight.

More concretely, consider a situation where there are a hundred workers. If everyone likes each other, then my condition is satisfied, while the GSC condition is not satisfied since the GSC condition allows for complements only across (two) groups of workers. Suppose instead that at the same firm there is one specific worker whom every other worker does not like, while the other 99 workers like each other. In this case, while the GSC condition is not satisfied, ours is satisfied. Suppose instead that we have two specific workers whom the other workers dislike. Our condition is satisfied as long as these two workers like each other, but the GSC is not satisfied. The same story goes as we increase the number of disliked workers, as long as these workers like each other. Moreover, in BQP preferences, while the GSC condition has restrictions on the range of complementarities/substitutabilities, our condition does not impose any restriction on the range.

We later show by examples that our condition is also necessary to guarantee the existence of a stable matching. Although our condition is *not* necessary and sufficient for existence, we found it relatively easy to conduct some simulations and generate an example that violates our condition and has no stable matching.

With that said, we humbly note that like the conventional conditions, our condition is rarely satisfied in a *generic* case where complementarities/substitutabilities are randomly determined. Suppose we have ten workers and five firms, and every total pairwise surplus b_{ij}^k is drawn from a random uniform distribution $U[-1, 1]$. Since the probability of $b_{ij}^k = 0$ is zero, the graph of total pairwise surplus is complete for every firm, meaning that the graph has

556,014 cycles for each firm with the ten workers. Then, even when the sign-consistency condition is satisfied in the economy, the sign-balance condition is hardly satisfied by chance. As in Table 1, our simulation with 10,000 draws shows that while theoretically, there is a tiny probability that the condition is satisfied, the condition was not satisfied even a single time. Note that the chance that conventional conditions such as the gross substitutes (GS) and GSC conditions are satisfied under the same setting is *precisely zero* if there are more than four workers in the economy. This is because of the requirement they have on numerical values of complementarities/substitutabilities, which will be discussed in Section 3.

On the other hand, we also point out that our condition allows for a new class of interesting settings where within the same group, agents like each other and across groups, they dislike each other as mentioned above. Note that a situation of this type violates both the GS and GSC conditions. Also note that our condition is a generalization of an ancient proverb and a basis of U.S. diplomatic policies[10], *the enemy of my enemy is my friend.* Our condition allows for this new set of interesting settings.

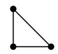
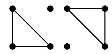
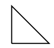
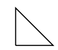
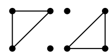
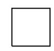
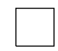
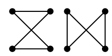
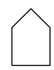
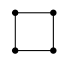
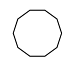
Next, we review the papers closest to ours. Then, after introducing our formal model environments, we start with pointing out the numerical restrictions on substitutabilities caused by the conventional conditions. Following the demonstration, we provide our results and make concluding remarks.

1.1 The Most Closely Related Literature

The closest papers to ours are Candogan et al. (2015), Candogan et al. (2018), Nguyen and Vohra (2018), and Baldwin and Klemperer (forthcoming). Nguyen and Vohra (2018)

[10]See, e.g., https://washingtonsblog.com/2014/09/americas-strategy-failing-world-complex-use-enemy-enemy-friend-strategy.html.

Table 1: Chances that our condition holds compared to GS and GSC when the BQP preferences and sign-consistency assumptions are satisfied

N **of workers**	2	3	4	5	$\cdots$	10
Number of cycles	0	1	7	37	$\cdots$	556,014
Components of cycles	$\emptyset$			 $\times$ 10	$\cdots$	 $\times$ 120
				 $\times$ 15	$\cdots$	 $\times$ 630
				 $\times$ 12	$\cdots$	$\vdots$
					$\cdots$	 $\times$ 181,440
% of our condition satisfies	100	50	11.8†	1.7†	$\cdots$	≈ 0
% of GS satisfies	*	*	0	0	$\cdots$	0
% of GSC satisfies	*	*	0	0	$\cdots$	0

†: These figures are obtained by a Monte Carlo simulation with 10,000 draws where every total pairwise surplus b_{ij}^k is drawn from a random uniform distribution $U[-1, 1]$.
$*$: We were not able to obtain what restrictions the GS and GSC conditions have when $N \leq 3$. On the other hand, we were able to generate an example that does not satisfy the GSC condition even though three workers (goods) could be divided into two "groups," within which b_{ij}^k is all negative and across which b_{ij}^k is all positive. Thus, there must be some numerical restrictions even when $N \leq 3$, though we could not derive them in this paper.

study two-sided matching market with couples in the presence of capacity constraints, while allowing for general preferences in the context of nontransferable utility. In the absence of utility transfer among agents, their setting inevitably encounters potential emptiness of the core set, and they overcome it by (possibly) minimally perturbing the capacity constraints. While their findings and algorithm are extremely powerful, the direct application of their results to transferable utility settings may produce excessively stringent outcomes[11].

[11]In fact, we can find an example where our findings do not work out in non-transferable utility settings.

Baldwin and Klemperer (forthcoming) provide a novel and powerful characterization of classes of valuations that result in Walrasian equilibria. In the sphere of their demand types, they provide necessary and sufficient conditions for such an equilibrium to exist. Note that valuations in our model are *not* in their classes of valuations, as demonstrated by Example 3.2 of the Supplemental Appendix in Candogan et al. (2015). Since the tree valuation class in Candogan et al. (2015) is a subset of our sign-balance valuation class, our model is also not in the class of valuations covered by Baldwin and Klemperer (forthcoming). Thus, none of their results and techniques are directly applicable to our model.

Candogan et al. (2015) provide necessary and sufficient conditions on agents' valuations to *guarantee* the existence of a Walrasian equilibrium in a one-sided matching market (or one auctioneer and many bidders with multiple items)[12]. They also exploit the BQP preferences and employ similar proof strategies for existence results. Furthermore, their conditions, the sign-consistency and tree structure on agents' valuation graph for a bundle of commodities, are similar to our conditions. We employ the same sign-consistency assumption, while we expand their tree valuation graph restriction to a sign-balance valuation graph that is a superset of theirs. For instance, the tree condition requires that if there are three goods A, B, and C, and non-zero complementarity/substitutability value attached to $\{A, B\}$ pair and $\{A, C\}$ pair, then there cannot be a non-zero complementarity/substitutability value attached to $\{B, C\}$ pair to

[12]To avoid readers' confusion, while Candogan et al. (2015) state "we establish that the sign-consistency and tree graph assumptions are necessary and sufficient for our existence results," by showing examples where violating one of the assumptions can lead to non-existence of a Walrasian equilibrium, these assumptions are, strictly speaking, not technically necessary since there can be many instances without the assumptions to have a Walrasian economy. So what they really mean is that violating one of their conditions *can* lead to a lack of a Walrasian equilibrium.

ensure the absence of a cycle in the entire graph. On the other hand, our sign-balance condition allows for a cycle (and a tree of course), while it restricts the structure of each cycle.

Aside from the difference in the domain of valuation, there are three major differences between their model and ours. One is that they study a one-sided matching model, whereas we study a two-sided matching model. This difference is crucial since the presence of one more side adds a lot more space for strategic actions by agents on both sides and thus makes much more opaque the nonemptiness of the core[13]. In the (combinatorial) auction literature, by the nature of the market, one-sided matching is a natural model to adopt and explore, whereas in the economics literature of matching, two-sided matching environments such as the school choice settings and labor markets are not uncommon.

Another difference is that Candogan et al. (2015) focus on welfare-maximizing Walrasian equilibria, while we focus on the core. In the presence of complementarities, as implied in Shapley and Scarf (1974), a Walrasian equilibrium may not have the core property. In two-sided matching markets, the literature has paid greater attention to the possibility of a blocking coalition formation and thus the core property. Therefore, in our two-sided matching settings, we believe that the core is the most suitable property to analyze.

Last but not least, Candogan et al. (2015) restrict the pricing scheme to be Walrasian pricing–i.e., the price of each item has to be non-discriminatory and be at the individual item level without any discounts or markups for a pair of items with bundle selling–, while we relax the pricing scheme to the

[13]For instance, it is well-known that Tom Brady intentionally reduces his salary so that Patriots can afford better teammates for him (https://brobible.com/sports/article/tom-brady-pay-cuts-help-patriots/).

so-called *agent-specific graphical pricing* rule (Candogan et al. (2018)). By such a restriction, Candogan et al. (2015) successfully show by example (and later theoretically with the series-parallel graph[14] valuation class in Candogan et al. (2018)) that their tree-valuation conditions are not just sufficient but also necessary to guarantee a Walrasian equilibrium. In contrast, our paper uses the more general class of pricing to find more relaxed conditions on the complementarity and substitutability structure. Note that our results are not covered by the results in Candogan et al. (2018), since their focus is to find conditions for an efficiently-computable pricing scheme at a pricing equilibrium in one-sided matching to exist, while our focus is to explore the most general sign structure of agents' valuations at a stable two-sided matching.

2 Environment

We follow the notations and terminologies in Liu et al. (2014) [15]. On one side, there is a finite set of workers, I, with an individual worker, $i \in I$. On the other side, there is also a finite set of firms, K, with an individual firm $k \in K$. A firm-worker (or a group of workers) match generates value. Following Liu et al. (2014), we take as primitive the match value each agent obtains without any payments between the agents and call it *premuneration value*[16]. Notationally,

[14] "A graph is a series-parallel graph if it can be obtained from a tree network by repeatedly adding an edge in parallel to an existing one or by replacing an edge by a path" (Definition 3.4 of Candogan et al. (2018)).

[15] Our work with incomplete information is available upon request

[16] Liu et al. (2014) write, "the firms premuneration value may include the net output produced by the worker with whom the firm is matched, the cost of the unemployment insurance premiums the firm must pay, and (depending on the legal environment) the value of any patents secured as a result of the workers activities. The workers premuneration value may include the value of the human capital the worker accumulates while working with the firm, the value of contacts the worker makes in the course of his job, and (again depending on the legal environment) the value of any patents secured as a result of the workers activities."

a match between worker i and firm k engenders the worker premuneration value $\nu_{i,k} \in \mathbb{R}$ and firm premuneration value $\phi_{i,k} \in \mathbb{R}$.

Furthermore, a match between firm k and a pair of workers $i, j \in I$ is assumed to give rise to pairwise complementarities/substitutabilities q_{ij}^k for firm k and d_{ij}^k for both workers i and j. These complementarities/substitutabilities are symmetric–i.e., $q_{ij}^k = q_{ji}^k$ and $d_{ij}^k = d_{ji}^k$. A match between firm k and a set of workers $S \subseteq I$ produces total surplus of $\sum_{i \in S} \phi_{i,k} + \sum_{i \neq j : i,j \in S} q_{ij}^k$ for firm k and $\nu_{i,k} + \sum_{j \neq i : i,j \in S} d_{ij}^k$ for every worker i in S. By $a_i^k = \phi_{i,k} + \nu_{i,k}$, we denote *total individual surplus*, and by $b_{ij}^k = q_{ij}^k + d_{ij}^k$ we denote *total pairwise surplus.* We assume $b_{ii}^k = 0$ since it is captured by total individual surplus. Adding up all the values, we define $V(T) = \sum_{i,k} a_i^k + \sum_{i \neq j,k} b_{ij}^k$ as *total surplus valuation* for some proper subset of all the agents $T \subseteq I \cup K$. We assume that all the aforementioned values are zero under no match.

Note that as stated and assumed in Candogan et al. (2015), the literature makes a standard assumption that such a total surplus value function is monotonic for every agent. This means that the more members a match contains, the higher the total surplus value will be. This assumption is to ensure that every worker (object in the auction literature) is matched to some firm to clear the market. Rather than making such an assumption, we assume that the labor market has firm $\emptyset$ (or so-called home) which hires any unmatched worker at zero price. This firm represents unemployment in the labor market.

The advantage of this assumption is not just to eliminate the monotonicity assumption but also to relax another assumption made for individual values in Candogan et al. (2015) and Candogan et al. (2018). They assume that every value generated by an individual agent-item match is nonnegative. While this seems innocuous in settings like auctions, this may not be a desirable assumption to make in the labor market, given that workers could harm firms by suing them, being a bad coworker and impeding teamwork, hurting firm reputation with illegal activities, and simply being an unproductive worker despite firms' capacity constraints[17].

In addition to these value systems, we have a salary in our model, a means of transferring utility from one agent to another. We denote by p_i^k a salary offered by firm k to worker i. With the aforementioned components, we define payoff functions of both parties in the following way. Given a match between a bundle of workers $S \subseteq I$ and firm k, the firm's payoff is $\pi_k := \sum_{i \in S} \phi_{i,k} + \sum_{i \neq j : i,j \in S} q_{ij}^k - \sum_{i \in S} p_i^k$, while for $i \in S$, worker i's payoff is $\pi_i := \nu_{i,k} + p_i^k + \sum_{j \neq i : i,j \in S} d_{ij}^k$. A *matching* is defined as a function $\mu : 2^I \to K \cup \{\emptyset\}$. Notice that it is not one-to-one. $\mu^{-1}(k)$ spits out a set of workers S assigned to firm k.

Finally, we denote a vector of payment $\mathbf{p}$ that lies in $\mathbb{R}^{2^I \times K}$ and specifies a salary transfer amount for every combination of a subset of a firm and group of workers. An *allocation* $(\mu, \mathbf{p})$ consists of a matching function μ and a salary vector $\mathbf{p}$.

2.1 Stability

We follow a standard definition of individual rationality and stability with transferable utility. A matching is *individually rational* if each agent gets at least as much as what he can get from the outside option of remaining unmatched, which is normalized to be 0.

[17]And the federal minimum wage may lead to negative net gains from hiring an unproductive worker.

Definition 2.1. An allocation (μ, $\mathbf{p}$) is *individually rational* if

$$\nu_{i,\mu_i} + p_i^{\mu(S)} + \sum_{j \neq i: i,j \in \mu^{-1}(S)} d_{ij}^{\mu(S)} \geq 0 \quad \forall i \in S \subseteq I, \forall S \subseteq I \tag{1}$$

and

$$\sum_{i \in \mu^{-1}(k)} \phi_{i,k} + \sum_{i,j \in \mu^{-1}(k)} q_{ij}^k - \sum_{i \in \mu^{-1}(k)} p_i^k \geq 0 \quad \forall k \in K \tag{2}$$

With individual rationality, we are ready to introduce the stability definition.

Definition 2.2. An allocation (μ, $\mathbf{p}$) is *stable* if it is individually rational, and there is no other allocation $(\hat{\mu}, \hat{\mathbf{p}})$ such that for some coalition $(S', k') = (\hat{\mu}^{-1}(k'), \hat{\mu}(S'))$ that may or may not be equal to $(S, k) = (\mu^{-1}(k), \mu(S))$,

$$\sum_{i \in S'} \phi_{i,k'} + \sum_{i \neq j: i,j \in S'} q_{ij}^{k'} - \sum_{i \in S'} \hat{p}_i^{k'} > \sum_{i \in S} \phi_{i,k} + \sum_{i,j \in S} q_{ij}^k - \sum_{i \in S} p_i^k \tag{3}$$

and for some $i \in S'$,

$$\nu_{i,k'} + \hat{p}_i^{k'} + \sum_{j \neq i: i,j \in S'} d_{ij}^{k'} > \nu_{i,k} + p_i^k + \sum_{j \neq i: i,j \in S} d_{ij}^k. \tag{4}$$

2.2 Core

Now, we shall define the core of our cooperative game below.

Definition 2.3. Denote by (N, V) a cooperative game with transferable utility, where $N = I \cup K$ and V is valuation from forming a coalition–i.e., $V(T) = \sum_{i,k \in T \subseteq N} a_i^k + \sum_{i,j,k \in T \subseteq N} b_{ij}^k$ for any $T \subseteq N$. Then, the core is a set of *imputations* $\pi \in \mathbb{R}^N$ satisfying

(i) Efficiency: $\sum_{i \in N} \pi_i = V(N)$

(ii) Coalitional Rationality: $\sum_{i \in T} \pi_i \geq V(T)$

Notice that any point in the core is a stable matching by definition.

3 Consequences of Conventional Conditions

Here, we discuss the consequence of the conventional substitutability and complementarity conditions with BQP preferences. We first provide general results and introduce a simple example. Before delving into the results, let us touch upon the definitions of various substitutability conditions. Many of these substitutability conditions with different settings such as the GS condition from Kelso and Crawford (1982) and the single-improvement (SI) condition from Gul et al. (1999) are all equivalent in unit demand (Tamura (2004))[18]. This is because all these substitutability conditions come down to the so-called $M^\natural$ concavity condition, which will be discussed in details in Appendix A. For this reason, we shall only focus on the GS condition since all the other conditions are equivalent when put in contexts. To define the GS condition, we introduce the demand correspondence for firm k and price vector $p^k \in \mathbb{R}^{2^I}$:

$$D\left(k, p^k\right) = \arg\max_S \left\{ \sum_{i \in S} \phi_{i,k} + \sum_{i \neq j : i,j \in S} q_{ij}^k - \sum_{i \in S} p_i^k \right\}.$$

Also, let $L^k(S) \equiv \left\{ i | i \in S \text{ and } p_i^k = \hat{p}_i^k \right\}$ for two different price vectors $p^k, \hat{p}^k \in \mathbb{R}^{2^I}$. We say that firm k's payoff function satisfies the GS condition if it satisfies

[18]See, also, Shioura and Tamura (2015). Furthermore, if agents can lie on either side of the two-sided economy for different transactions of trade, even the GSC condition from Sun and Yang (2006) is equivalent to these substitutability conditions and the so-called full-substitutability condition (Hatfield et al. (2015)).

the following:

(GS) for every firm k, if $S \in D(k, p^k)$ and $p^k \leqq \hat{p}^k$,

then there exists $S' \in D(k, \hat{p}^k)$ such that $L^k(S) \subseteq S'$.

In words, the GS condition means that a firm still wants to hire workers whose salaries remain unchanged after the salaries of the other workers hired at the firm increase.

Now, we are ready to provide Proposition 1, an extension of Murota (2003), Murota and Shioura (2004), and Iwamasa (2018). Proposition 1 portrays the restrictions on the range of total pairwise surplus, b_{ij}^k, by imposing the GS condition on BQP preferences. Note that Proposition 1 is a new result since the domain in Murota (2003) and Murota and Shioura (2004) is either n-dimensional reals or integers, including negatives. This is crucial since allowing for negatives restricts individual goods' valuation to be all non-positive (i.e., $a_i^k \leq 0$ for every (i, k) pair), making problems uninteresting to start with[19]. The economics literature focuses on assigning different types of agents to different types of goods or firms with positive indivisible amounts, and therefore, the hypercube domain is a natural setting in two-sided matching problems. Iwamasa (2018) derives the conditions for M-convex obective functions of BQP preferences within this domain but not for $M^\natural$ concave quadratic functions. We provide our proof in Appendix A.

Proposition 1. *For an economy with more than four workers and BQP preferences, firm k's payoff function satisfies the (GS) condition if and only if for*

[19]See Theorem 5.3 of Murota and Shioura (2004) and Murota (2003) page 139, Proposition 6.8, equation (6.28) for $M^\natural$-*convex* quadratic functions. Notice that the condition includes a case with $(i = j)$

every distinct worker, $i, j, o, l \in I$*,*

$$b_{ij}^k + b_{ol}^k \leq \max\left\{b_{io}^k + b_{jl}^k, b_{il}^k + b_{jo}^k\right\}, \tag{5}$$

$$b_{ij}^k \leq \max\{b_{io}^k, b_{jo}^k\}, \tag{6}$$

$$b_{ij}^k \leq 0. \tag{7}$$

Just to give an idea of what (5) and (6) together mean, consider an economy with four workers (or items) (n = 4). Let $b_{ij}^k = \omega_{\max(i,j)}^k \quad (1 \leq i, j \leq n)$ for $i \neq j$ and for some $\omega_1^k \leq \omega_2^k \leq ... \leq \omega_n^k$ that satisfies (5) above. Then, $B_k = \left(b_{ij}^k\right)_{i,j=1}^n$ would look like the following:

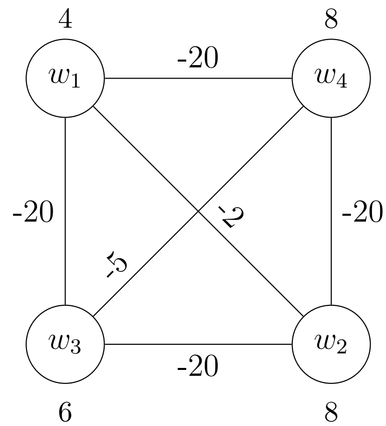
$$B_k = \begin{bmatrix} 0 & \omega_2^k & \omega_3^k & \omega_4^k \\ \omega_2^k & 0 & \omega_3^k & \omega_4^k \\ \omega_3^k & \omega_3^k & 0 & \omega_4^k \\ \omega_4^k & \omega_4^k & \omega_4^k & 0 \end{bmatrix}$$

On the ohter hand, the meaning of (6) is straightforward; pick any triples i, j, k. Then, if one of substitutability value, say b_{ij}^k is (weakly) larger than another b_{ik}^k, then b_{ik}^k has to equal b_{jk}^k so that the GS condition is satisfied. Conditions (5) and (6) are quite demanding, and thus violating any of the two conditions is easy, even when all the substitutability values are negative, or when (7) holds. Conditions (5) and (6) are the reasons for the figures in Table 1.

Consider a situation where a firm is hoping to hire some workers and has four choices: w_1, w_2, w_3, w_4[20]. For simplicity, suppose workers only care about

[20] A more chewable example can be a consumer hoping to buy some beverages at a store with two kinds of tea (w_1 and w_2) and two kinds of soda (w_3 and w_4).

Figure 1: An example of preference that violates GS



their salaries. This is innocuous because after all, due to transferability of utility, we can reformulate the problem this way. Say the firm has preferences described in Figure 1. This firm obtains value amounts written above each node from hiring individual workers–i.e., $\pi(w_1) = 4, \pi(w_2) = 8, \pi(w_3) = 6, \pi(w_4) = 8$, while she obtains values juxtaposed to each edge from hiring a pair of workers–i.e., $\pi(\{w_1, w_3\}) = -20, \pi(\{w_1, w_4\}) = -20, \pi(\{w_1, w_2\}) = -2, \pi(\{w_2, w_3\}) = -20$, $\pi(\{w_2, w_4\}) = -20, \pi(\{w_3, w_4\}) = -5$. Suppose all the prices $p_{w_1} = p_{w_2} = p_{w_3} = p_{w_4} = 1$ initially. Then, this consumer's demand correspondence is $\{w_1, w_2\}$. Now, suppose the salary of w_1 increases from 1 to 3 ($1 = p_{w_1} < p'_{w_1} = 3$). The GS condition requires her new demand correspondence after this price change to include w_2 since her wage has remained unchanged. However, notice that her new demand correspondence is $\{w_3, w_4\}$.

While the GS condition guarantees existence even under a considerably more general preference than a BQP preference, this example demonstrates the numerical restrictions of the GS condition under the restricted preferences. The GSC condition has the same restriction within the same group to which a good belongs. One may think that it is straightforward to obtain the equivalent condition of Proposition 1 for the GSC condition, which turns out to be not[21]. While our condition has no such restriction on the range of complementarity values, we do not boldly claim that our condition is more feasible in a generic case, as discusssed in Section 1. Rather, the importance of our results is that our condition allows for a new class of interesting settings that the GS and GSC conditions cannot tolerate. These new settings are particularly applicable to labor markets as mentioned in Section 1.

4 Existence

Before introducing our condition, we need to introduce a graph corresponding to total pairwise surplus, b_{ij}^k . Let $G = (I, E)$ be the value graph associated with the graphical valuations of worker complementarity, with a node set I and with an edge that exists whenever for any $i, j \in I$, $b_{ij}^k \neq 0$. By the sign-consistency assumption, we can divide the edges into positive ($b_{ij}^k \geq 0$) and negative ($b_{ij}^k < 0$) groups. We denote these two sets by $E^+ = \{(i,j) : b_{ij}^k \geq 0\}$ and $E^- = \{(i,j) : b_{ij}^k < 0\}$. Note that when we say a cycle, we mean a simple cycle that is a circuit[22] in which no vertex except the first one appears more than once.

We first introduce the so-called sign-consistency assumption introduced by Candogan et al. (2015). The idea is that roughly speaking, all firms agree on which pair of workers go well (poorly) together. The sign of a total pairwise

[21]One may think that since by definition, function f that satisfies gross complements can be obtained by taking the negative of another function $-f$ that satisfies the GS condition, one may think that the GSC condition has to satisfy conditions from Proposition 1 within the same group to which a good belongs and has to satisfy the negative of these conditions across (two) groups. However, this may not be true since a portion of those goods that are gross complements are simultaneously gross substitutes to a subset of the portion.

[22]A circuit is a path of edges and nodes in which the starting node is reachable from itself.

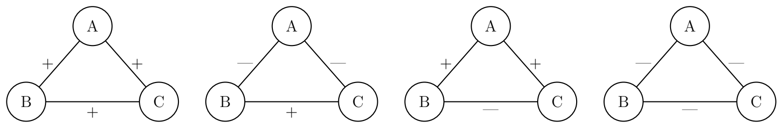
surplus b_{ij}^k can be either negative or positive as long as it is consistent across firms. Notice that our model is flexible enough to tolerate the sign inconsistency for the components of total pairwise surplus. For example, q_{ij}^k can be positive at firm k but negative at firm k' as long as the sign of b_{ij}^k is consistent across firms. In other words, even if a pair of workers are complements at one firm and happen to be substitutes at another firm, as long as these workers are happy by working together (e.g., a couple) at the latter firm to generate sufficient utility to offset the negatives, then b_{ij}^k is positive at both firms. Conversely, a pair of workers can like working together at one firm but dislike working together at another firm, as long as they together generate enough joint production for the latter firm to make b_{ij}^k positive.

Assumption 4.1. *(Sign Consistency). For some* $(i, j) \in E$ *and* $k \in K$*, if* $b_{ij}^k > 0$*, then* $b_{ij}^l \geq 0$ *for all* $l \in K$*, and similarly, if* $b_{ij}^k < 0$*, then* $b_{ij}^l \leq 0$ *for all* $l \in K$*.*

Next, we introduce the so-called sign-balance assumption. Colloquially, the condition requires that the enemy of my enemy is my friend. The important property of a sign-balance graph is so-called *clusterability* (Cartwright and Harary (1956)); one can regroup the nodes of the graph into groups within which $b_{ij}^k > 0$ or $b_{ij}^k = 0$ and across which $b_{ij}^k < 0$. This seems qualitatively opposite of the GSC condition, although the two conditions are mathematically not opposite due to the numerical restrictions under BQP preferences. Figure 2 shows examples of a sign-balance graph. The two graphs at the left are sign balanced, while the other two are not.

Assumption 4.2. *(Sign Balance). Let* $G = (I, E)$ *be the value graph associated with the graphical valuations of worker complementarities. We assume*

Figure 2: Examples of a cycle in balanced and unbalanced graphs



The two graphs at the left are balanced, while the other graphs are unbalanced

that G is a sign-balance graph–i.e., any simple cycle in the graph contains an even number of negative edges.

Now, we are ready to state Theorem 1 on the existence of a stable matching. Our proof exploits the primal-dual relation between welfare-maximizing solutions and the core. In particular, we first show that the following quadratic program (QP1) has an integer-value solution:

$$\begin{aligned} \text{maximize} & \sum_{k \in K} \sum_{i \in I} a_i^k x_i^k + \sum_{k \in K} \sum_{i \neq j} b_{ij}^k x_i^k x_j^k \\ \text{subject to} & \sum_{k \in K} x_i^k \leq 1 \;\; \forall i \in I \\ & 0 \leq x_i^k \leq 1 \;\; \forall i \in I, \forall k \in K, \end{aligned}$$

where $x_i^k = 1$ if worker i is assigned to firm k, $0 < x_i^k < 1$ if *a fraction* of worker i is matched to firm k, and $x_i^k = 0$ if worker i is not assigned to firm k. The constraint, $\sum_k x_i^k \leq 1$, ensures that worker i is assigned no more than once.

We prove the existence of an integral solution by extending the version of the proof for the tree-valuation graph from Candogan et al. (2015) written in one of Vohra's blog posts (2014)[23]. His proof uses induction, in particular

[23]https://theoryclass.wordpress.com/2014/02/10/combinatorial-auctions-and-binary-quadratic-valuations-postscript/

showing that an extreme point in the polyhedron of the welfare-maximizing problem formulated in the linear programming manner is integral for every natural number of the cardinality of the maximal connected components of the worker valuation graph after deleting negative edges. Exploiting the tree structure, he divides the graph into one connected component and the complement of the component, which allows him to formulate the original problem as the convex combination of the two parts of the graph, both of which have integral solutions.

To apply his clever method to our model, there are two points to expand on his version of the proof, due to the difference in graph structures. First, for the n cardinality case, unlike the tree-valuation graph, our graph may not have a component of the maximal connected components that has a node with exactly one negative edge to a node in one of the other maximal connected components; rather, a node in such a component can be incident to multiple negative edges. Furthermore, within this component, there may be multiple nodes that are connected to other components with negative edges. We provide our proof to deal with these two points at Appendix A.

Lemma 1. *Let Assumption 4.1 and 4.2 hold. Then, (QP1) has an integral solution.*

Note that as implied in Shapley and Scarf (1974), this is not enough to prove that the welfare-maximizing allocation does actually have the core property. With Lemma 1, our main theorem can be obtained by linearizing (QP1) and applying the primal-dual approach of a linear programming framework. There are two components in the proof to highlight.

One is that our proof uses an equivalent formulation of the original primal problem whose dual does not immediately correspond to the core. Our technique provides a way for future research to find a point in the core when researchers study matching with complementarities that are beyond the existing class of complementarities. Second, in the course of showing that the dual of the equivalent formulation indeed corresponds to the core, we extend the many-buyer-with-one-seller setting of Theorem 5.3.1 from Vohra (2011) to many buyers with many sellers who can collude to sell items. Importantly, our condition allows us *not* to specify a particular bargaining structure among agents to eliminate a blocking coalition.

Theorem 1. *Let Assumption 4.1 and 4.2 hold. Then, an integral solution to (QP1) lies in the core.*

Proof. We first introduce a new variable, z_{ij}^k, to linearize the quadratic terms in (QP1):

$$\begin{aligned}
\text{maximize} & \sum_{k\in K}\sum_{i\in I} a_i^k x_i^k + \sum_{k\in K}\sum_{(i,j)\in E^+\cup E^-} b_{ij}^k z_{ij}^k \\
\text{subject to} & \sum_{k\in K} x_i^k \le 1 \;\; \forall i \in I \\
& z_{ij}^k \le x_i^k, x_j^k \;\; \forall k \in K, (i,j) \in E^+ \\
& z_{ij}^k \ge x_i^k + x_j^k - 1 \;\; \forall k \in K, (i,j) \in E^- \\
& x_i^k, z_{ij}^k \ge 0 \;\; \forall i,j \in I, \forall k \in K
\end{aligned}$$

Note that we can formulate this way, due to Bertsimas et al. (1999). We call this relaxed formulation (LP1). We claim that we can reformulate (LP1) into the following equivalent form (P1) with the corresponding dual (DP1):

P1

$$
\begin{aligned}
V(N) = \max \sum_{k\in K}\sum_{S\subseteq I} v_k\,(S)\,x_k(S) \\
\text{subject to} \sum_{S\ni i}\sum_{k\in K} x_k(S) \leq 1 \;\; \forall i\in I \\
\sum_{S\subseteq I} x_k(S) \leq 1 \;\; \forall k\in K \\
x_k(S) \geq 0 \;\; \forall S\subseteq I, \forall k\in K,
\end{aligned}
$$

DP1

$$
\begin{aligned}
Z_1(N) = \min \sum_{k\in K} \pi^k + \sum_{i\in I}\pi_i \\
\text{subject to } \pi^k + \sum_{i\in S}\pi_i \geq v_k(S) \\
\forall S\subseteq I \\
\pi^k \geq 0 \;\; \forall k\in K \\
\pi_i \geq 0 \;\; \forall i\in I
\end{aligned}
$$

where $v_k(S) = \sum_{i\in S} a_i^k + \sum_{i,j\in S} b_{ij}^k$ and $x_k(S)$ indicates an integral or fraction of a set of workers S assigned to firm k. The first constraint ensures that each worker is assigned at most once. The second ensures that no firm receives more than one subset of workers. Call this formulation (P1). The reason why (LP1) is equivalent to (P1) is that we have one-to-one mapping between decision variables (x, z) and $x(S)$. Thus, for some (x, z), there always exists a vector $x(S)$ such that

$$
\sum_{k\in K}\sum_{S\subseteq I} v_k\,(S)\,x_k(S) = \sum_{k\in K}\sum_{i\in I} a_i^k x_i^k + \sum_{k\in K}\sum_{(i,j)\in E^+\cup E^-} b_{ij}^k z_{ij}^k,
$$

and vice versa. By the same token, it is trivial to show that a point in the feasible set of (LP1) lies in that of (P1), and vice versa. It is important to keep in mind that one formulation is *not* an extended formulation of the other, which is the reason why the logic below works.

By the integrality of the solution to (QP1) and thus (LP1), the dual is exact. Due to the equivalence, the integral solution to (LP1), $(x^\star, z^\star)$, is also a solution to (P1). Now, take an optimal solution to (DP1), $(\pi^{\star})$, and consider a subset of workers and firms R. Denote by $(\pi^{\star}(R)) = (\{\pi^{k^{\star}}\}_R, \{\pi_i^{\star}\}_R)$ an optimal solution to the dual when restricted to a subset R. Now, we can compute the objective value of the dual for a subset of workers and firms R, $Z_1(R) = \sum_{k \in R} \pi^{k^{\star}} + \sum_{i \in R} \pi_i^{\star} \geq V(R)$ where $V(R)$ is the objective value of (P1) when restricted to R and the inequality is due to weak duality.

Now, by strong duality (coming from the integrality of a solution to (P1)),

$$\sum_{k \in K} \pi^{k^{\star}} + \sum_{i \in I} \pi_i^{\star} = V(N)$$

■

Furthermore, with the counter-examples that we provide at Section 5 and Appendix B, we can state that Assumption 4.1 and Assumption 4.2 are both necessary to *guarantee* the existence of a stable matching if agents have BQP preferences and if a worker valuation graph of *any* firm contains at least one cycle of length three (triangle). We think that the last added assumption of at least one triangle is rather weak since say if we have ten workers, there are 120 possible triangles at every firm.

Corollary 1. *Suppose agents have BQP preferences, 4.1 holds, and a worker valuation graph of any firm contains at least one cycle of length three (triangle). Then, the sign-balance conditions are necessary and sufficient to guarantee the existence of a stable matching.*

One may wonder if we can obtain a specific form of equilibrium prices at a stable matching. First of all, we note that since our model does not form a lattice, there is no tâtonnement process like the one in Sun and Yang (2009). As Candogan et al. (2015) show that the pricing rule in a sign-consistent tree valuation graph does not form a lattice, our valuation graph also does not have a lattice structure since a tree valuation graph class is a subclass of a sign-balance graph class. While it appears difficult for us to derive any specific form of supporting prices, we believe that in general such equilibrium prices have to follow a non-Walrasian pricing rule–i.e., different firms offer different salaries to a specific subset of workers–, due to Candogan et al. (2018)[24]. While uniform (Walrasian) prices make sense in settings like auctions, we believe that firm-specific salaries are commonly practiced and more realistic in labor markets than Walrasian prices.

5 Intuition and Numerical Examples

5.1 Intuition

Suppose we have worker A, B, and C and firm 1 and 2. Suppose we hold an auction with second-price sealed bids and ask every coalition of a firm and subset of workers how much they will jointly pay to be matched together. We let those who pay the highest be matched together and remove these agents from the auction. At this point, we give an opportunity for the remaining

[24]There are two components in such pricing rules. One is anonymity. A Walrasian pricing rule (i.e., a pricing rule in a Walrasian equilibrium) is defined in a way that the price offered to firm k for a set of workers S can be expressed as the sum of individual wages for each worker in S and is equal to $p^k(S) = \sum_{i\in S} p_i$, where $\{p_i\}$ indicates anonymous worker price/salary (i.e., every firm faces the same, non-discriminatory market salary amount for each worker).

The other component is the inclusion of discounts/markups for a pair of goods. We call pricing *graphical* if the transfer made from firm k to a set of workers S can be expressed as $p^k(S) = \sum_{i\in S} t_i^k + \sum_{i\neq j:i,j\in S} p_{ij}^k$. In particular, we call a pricing rule *agent-specific graphical pricing* if $p^k(S) = \sum_{i\in S} p_i^k + \sum_{i\neq j:i,j\in S} p_{ij}^k$ and *anonymous graphical pricing* if $p^k(S) = \sum_{i\in S} p_i + \sum_{i\neq j:i,j\in S} p_{ij}$. However, the results fromCandogan et al. (2018) are all based upon a one-sided economy with one seller and thus are not directly applicable to our model. In fact, one can verify that there is no pricing rule of such forms that induces a stable matching for the two-sided economy in Figure 4.

agents to form a coalition with the removed agents to block the matching. Then, we repeat the same process until we are left with no agent.

This process terminates only at the first two cases of Figure 2–all positive complementarities and sign-balance cases (the farthest to the left and the second farthest). For simplicity, suppose workers only care about their salaries. This is innocuous because after all, due to transferability of utility, we can reformulate the problem this way. For the first case with all positives, we just need to find out which firm has the highest value on the combination of three workers and let the firm pays the value and get all the workers. As for the second case with two negatives and one positive, we need to discover which firm values $\{B, C\}$ together more and let that firm be matched with the two workers and let the other firm be matched with worker A.

Regarding the third case with one negative, both firms want worker A. If we can cut worker A into pieces, then we can let some pieces of A be matched with one firm and the rest of A be matched with the other firm. However, we assume that worker A is indivisible, so there will be a conflict of interest between the two firms. This is the source of instability in the case with an odd number of negatives.

A similar story can happen in the last case with three negatives. Say, the magnitude of negatives for $\{A, B\}, \{A, C\}$ is small, while that for $\{B, C\}$ is large. If the individual surplus of A is large enough to cancel the negatives, then the same conflict can happen between the two firms. Notice that this is not the case with two negatives; if the individual surplus of A is large to cancel the two negatives, then it is practically the same as the case with three positives and we just need to find a firm that has the highest values on the combination of the three workers.

5.2 Numerical Example

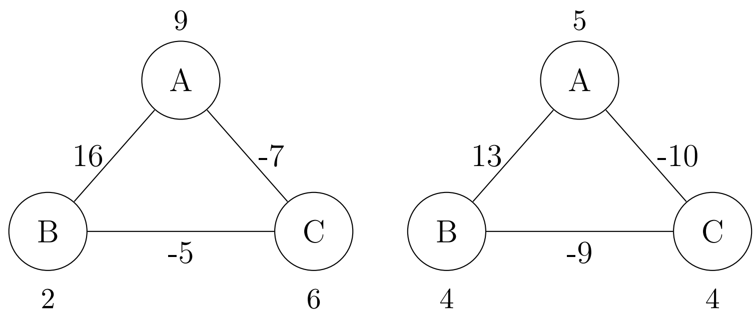
We focus on the second and third cases in Figure 2 to delineate our intuition numerically. Set valuation graphs as the one in Figure 3. The valuation graph at the left is for firm 1, and the one at the right is for firm 2. The number next to each node is total individual surplus, a_i^k; for example, firm 1 jointly produces a surplus of 9 with worker A, a surplus of 2 with worker B, and a surplus of 6 with worker C. Also, every total pairwise complementarity, b_{ij}^k is represented by the number juxtaposed to every edge in Figure 3; for instance, firm 2 and worker $\{A, B\}$ pair enjoy a total pairwise surplus of 13.

Suppose we conduct an auction with the aforementioned procedure. Settling down this auction is straightforward; find a firm that values a bundle of workers with positive complementarities ($\{A, B\}$) the highest, let the firm be matched with these workers and let the other firm be matched with C. In our example, worker A and B jointly sell themselves to firm 1 at the price of 18 (the sum of their total individual and pairwise surplus at firm 2 *minus* a surplus of 4 that worker C produces at firm 2), while worker A gets 9.5 (firm 2's total individual surplus with worker A plus half of the total pairwise surplus minus worker C's surplus 4) and worker B gets 8.5 (firm 2's total individual surplus with worker B plus half of the total pairwise surplus minus worker C's surplus 4)[25]. Worker C sells himself to firm 2 at the net wage of zero (for example, if $\nu_{C,f2} = -2$ and $\phi_{C,f2} = 6$, leading to $a_C^{f2} = 4$, then $p_C^{f2} = 2$). Notice that there is no blocking coalition here.

Now, suppose we have valuation graphs as in Figure 4 instead. Notice that there is no stable matching in this example. Just to illustrate how the infinite loop of blocking occurs, let us look at an arbitrary start of this loop. Suppose

[25]This way of dividing complementarity value is not unique by any means.

Figure 3: Example with a stable matching



The left graph is the valuation graph for firm 1 and the right graph is that for firm 2. In this case, we have a stable matching. The number right next to each node is total individual surplus, a_i^k; for example, firm 1 jointly produces a surplus of 9 with worker A, a surplus of 2 with worker B, and a surplus of 6 with worker C. Also, every total pairwise complementarity, b_{ij}^k is represented by the number juxtaposed to every edge; for instance, firm 2 and worker A and B pair enjoy a total pairwise surplus of 13.

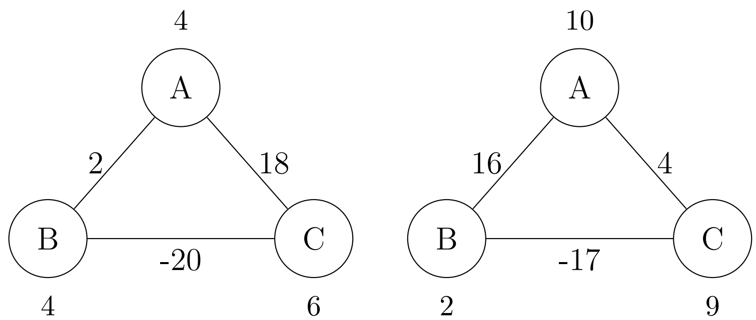
firm 1 gets worker A and C together, while firm 2 hires worker B. At a glance, this seems to be an efficient allocation and thus achieves a stable matching with the right pricing rule. Notice that if firm 1 pays worker A less than 26, then firm 2 forms a blocking coalition with worker A and B. So, firm 1 has to pay worker A 26, and pays worker C no more than 2 since otherwise, firm 1 would obtain a negative payoff. But then, firm 2 will leave worker B and form a blocking coalition with worker C, paying her any amount in (2, 7) (since he can get at most 2 from worker B).

Similar blocking processes will happen at any combination of allocation, and thus we can show this is an example with no stable matching.

6 Conclusion

Restricting preferences to BQP preferences, this paper (a) showed that the conventional sufficient conditions for the existence of a stable matching sub-

Figure 4: Example without a stable matching



This is a counter-example to the existence of a stable matching that violates the sign-balance condition. The left graph is the valuation graph for firm 1 and the right graph is that for firm 2.

stantially restrict the *range* of complementarities, (b) provided a new sufficient condition without any range restriction that is relevant to labor markets, and (c) established that under some minor assumption, our condition also is necessary to *guarantee* existence. We believe that our condition allows for a new class of interesting settings as described above. Whether in school, the workplace, a community, a country, or the globe, we tend to see that one joins a group of people with whom he or she finds comfortable, and across these groups, people fight.

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Appendices

A Proofs of Proposition 1 and Lemma 1

Proof of Proposition 1

To derive Proposition 1, we invoke Murota's equivalence theorem between a function that satisfies the GS condition and a $M^\natural$-concave function[26]. First, we introduce general definitions of *M-concave* functions following Iwamasa (2018).

Note that we do not follow section 5.2 of Murota and Shioura (2004) since their definition uses the domain to be either n-dimensional reals or integers, including negatives. This is crucial since allowing for negatives restricts individual goods' valuation to be all non-positive, making problems uninteresting to start with (see Theorem 5.3 of Murota and Shioura (2004) and Murota (2003) page 139, Proposition 6.8, equation (6.28). Notice that the condition includes a case with $(i = j)$). Furthermore, since the literature focuses on assigning different types of agents to different types of goods or firms with positive indivisible amounts, the domain to be hypercube is a natural setting in markets like labor markets.

Definition 1. A function f on $\{0,1\}^n$ is said to be *M-convex* if it satisfies the following generalization of matroid exchange axiom:

Exchange Axiom: For $x, y \in \operatorname{dom} f$ and $i \in \operatorname{supp}(x) \setminus \operatorname{supp}(y)$, there exists

[26] Murota (2003) says that $M^\natural$ read as "M natural", though some people read it as "M sharp".

$j \in \text{supp}(y) \setminus \text{supp}(x)$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j),$$

where $f := \{x \in \{0,1\}^n | f(x) \text{ takes a finite value}\}$ is the effective domain of $f, \text{supp}(x) := \{i | x_i = 1\}$ for $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$, and χ_i is the ith unit vector.

Now, we are ready to introduce the definition of a $M^\natural$-convex function in the domain of a hypercube. Note that we write $x(N) = \sum_i^n x(i)$ where $x(i)$ is the ith element of vector x.

Definition 2. A function f is called $M^\natural$*-convex* if function $\hat{f} : \{0,1\}^{|\hat{N}|} \to \mathbb{R}$ for $\hat{N} = \{0\} \cup N$ defined by

$$\hat{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(N)) \\ +\infty & (\text{ otherwise }) \end{cases} \qquad \left((x_0, x) \in \{0,1\}^{|\hat{N}|}\right)$$

is M-convex.

With this definition, we first introduce results from Iwamasa (2018). Just to avoid confusion, we use α instead of b to denote complementarity values. Also, we drop firm index since the GS condition is imposed on every firm.

Theorem 1. *(Theorem 1.1 in Iwamasa (2018)) Let $A = (\alpha_{ij})_{i,j=1}^n$ be a symmetric matrix, and suppose α_{ij} takes a finite value for all distinct $i, j \in N$. Then, a function $f : \{0,1\} \to \bar{\mathbb{R}}$ defined by*

$$f(x_1, x_2, \ldots, x_n) := \begin{cases} \sum_{i \in N} a_i x_i + \sum_{1 \leq i < j \leq n} \alpha_{ij} x_i x_j & \text{if } \sum_{i \in N} x_i = r \\ +\infty & \text{otherwise} \end{cases} \tag{8}$$

is M-convex if and only if

$$\alpha_{ij} + \alpha_{kl} \geq \min\{\alpha_{ik} + \alpha_{jl}, \alpha_{il} + \alpha_{jk}\}$$

holds for every distinct $i, j, k, l \in N$*.*

Now, we are ready to derive Proposition 1. Applying Definition 2 from this section, we define $\tilde{f}$ as

$$\tilde{f} = \frac{1}{2}(x_0, x)^T \tilde{A}(x_0, x),$$

where

$$\tilde{V} = \begin{bmatrix} \alpha_{00} & \alpha_0^T \\ \alpha_0 & A \end{bmatrix}$$

for some vector $\alpha_0 \in \mathbb{R}^n$. Note that using Theorem 1.1 from Iwamasa (2018) above, we know the corresponding function f is M-convex. Then, to derive conditions for coefficients in A, we need to check the conditions that make $\tilde{f}(x) = f(x)$. If $x_0 = -x(N) = -\sum_{i=1}^{n} x(i)$, then we have

$$\begin{bmatrix} -x(N) & x \end{bmatrix}^T \tilde{A} \begin{bmatrix} -x(N) & x \end{bmatrix} = x^T A x$$

That is,

$$x(N)^2 \alpha_{00} - 2x(N)\alpha_0^T x + x^T A x = x^T A x. \tag{9}$$

Then, we get

$$\alpha_{00} x(N) = 2\alpha_0^T x. \tag{10}$$

Expanding both sides, we get

$$\alpha_{00}x(1) + \alpha_{00}x(2) + ... + \alpha_{00}x(n) = 2(\alpha_{10}x(1) + \alpha_{20}x(2) + ... + \alpha_{n0}x(n)), \tag{11}$$

which implies $\alpha_0 0 = 2\alpha_{10} = 2\alpha_{20} = ... = 2\alpha_{n0}$. For $\tilde{f}$ to be M-convex, from Iwamasa (2018), we need, for $l = 0$, $\alpha_{ij} + \alpha_{k0} \geq \min\{\alpha_{ik} + \alpha_{j0}, \alpha_{i0} + \alpha_{jk}\}$, which implies

$$\alpha_{ij} \geq \min\{\alpha_{ik}, \alpha_{jk}\}. \tag{12}$$

Similarly, for $k = l = 0$,

$$\begin{aligned} \alpha_{ij} + \alpha_{00} &\geq \min\{\alpha_{i0} + \alpha_{j0}, \alpha_{i0} + \alpha_{j0}\} \\ &= \alpha_{i0} + \alpha_{j0}, \end{aligned} \tag{13}$$

which implies

$$\alpha_{ij} \geq 0 \quad \forall i \neq j. \tag{14}$$

The last quantifier $i \neq j$ comes from the fact that since the domain is $\{0,1\}^n$ hypercube, the quadratic terms becomes linear for $i = j$.

Since a function h is M-concave if $-h$ is M-convex, so is h to be $M^\natural$-concave. Note that Fujishige and Yang (2003) prove that $M^\natural$-concavity for a set function is equivalent to the GS condition. Thus, taking the negative of (12), (13), and (14), we get Proposition 1.

Proof of Lemma 1

As an acknowledgement, we note that the following proof to show an integral solution to the linear programming with a sign-balance graph below is based heavily on Rakesh Vohra's personal blog posts that demonstrate a simpler proof for an integral solution to a linear programming with a tree-valuation graph (no cycle) from Candogan et al. (2015)[27], while all the errors and mistakes are ours. Meanwhile, in the course of extending his proof to sign-balance graph cases, we think that we simplified a portion of the original proof even more by adopting a different formulation.

Proof. We start with (LP1) from the proof for Theorem 1. Let P be the polyhedron of feasible solutions to (LP1). The goal is to show the extreme points of P are integral, which implies there exists a feasible optimal solution that is integral. The way to do this is to use the fact that if the constraints matrix is totally unimodular, then the extreme points of P are integral.

Denote by $G+$ the same graph with G except that all the edges in E^- are removed, and denote by $\mathbf{C}$ the maximal connected components of $G+$. We will prove the statement by induction.

Let $|\mathbf{C}| = 1$. Since G is clusterable and can be divided into groups of positive edges even before deletion of negative edges, $|\mathbf{C}| = 1$ implies that G only had positive edges in the first place. This means that all the coefficients $b_{ij}^k \geq 0$. Then, we can formulate the problem as the following. Let $z_{ij}^k = \min\{x_i^k, x_j^k\}$. That is, $z_{ij}^k \leq x_i^k \; \forall i \in I, k \in K$ and $z_{ij}^k \leq x_j^k \; \forall j \neq i \in I, k \in K$. Then, for $b_{ij}^k \geq 0$, we can reformulate the given problem as

$$\begin{aligned} \text{maximize } & \sum_{k \in K} \sum_{i \in I} a_i^k x_i^k + \sum_{k \in K} \sum_{i<j} b_{ij}^k z_{ij}^k \\ \text{subject to } & z_{ij}^k - x_i^k \leq 0 \; \forall i \in I, k \in K \\ & z_{ij}^k - x_j^k \leq 0 \; \forall j \in I, k \in K \\ & x_i^k \leq 1 \; \forall i, k \\ & 0 \leq z_{ij}^k \; \forall i, j, k \end{aligned}$$

[27]https://theoryclass.wordpress.com/2014/02/10/combinatorial-auctions-and-binary-quadratic-valuations-postscript/

Take an extreme point solution so that the first and second constraint equations achieve equality between the left- and right-hand sides. With this solution, multiply the second constraint by -1. Then, every column involving firm k has at most one +1 and at most one -1 coefficient (one in ith row and the other in jth row), and therefore the constraint matrix is a network matrix. This implies that the constraint matrix is totally unimodular, which implies that an extreme point solution is integral.

Now, suppose the statement holds for $|\mathbf{C}| = n$ for some natural number n, and consider $|\mathbf{C}| = n + 1$ case. Let $(\bar{z}, \bar{x})$ be an optimal solution to (LP1). We can choose it to be an extreme point of the polyhedron P. Since $\mathbf{C}$ is clusterable, meaning that we can divide nodes into groups with only positive edges within the groups and with negative edges across groups, we can take $C \in \mathbf{C}$ that has a node with negative edge(s) connected to at least another group $C' \in \mathbf{C}$ (if there does not exist such C, then the same proof from $|\mathbf{C}| = 1$ case carries through). Let P' be the polyhedron restricted to the nodes of C and let Q be that restricted to the vertices of the complement of C. Then, consider any node p of C that is connected to a member of another group C', say q, and another group C'', say q'. Note that if there is no such q', then the proof becomes easier. In that case, we can simply omit the part of the proof involving q' below.

We know that the sign of the edge (p, q) and (p, q') is negative. By the induction hypothesis, both P' and Q are integral polyhedrons. Note that every extreme point of P' (Q) assigns a node of C (the complement of C) to a certain firm k. Now, let $X_1, ..., X_a$ be the set of extreme points of P'. By $X^p_{rk} = 1$, we mean that in extreme points of X_r, node p is assigned to a firm k and 0 otherwise. By the same token, $Y^q_{rk} = 1$ if in extreme points of Y_r, node q is assigned to a firm k. Let $v(.)$ be the objective value of any extreme point

X_r or Y_r.

Since a polyhedron is convex, we can express $(\bar{z}, \bar{x})$ restricted to P' as $\sum_r \lambda_r X_r$ while $(\bar{z}, \bar{x})$ restricted to Q as $= \sum_r \zeta_r Y_r$. Let E_C^- as the set of negative edges restricted to those involving the vertices in C. Since λ_r is just a number between 0 and 1, we can let $\lambda_r = \sum_{p:(p,q)\in E_C^-} \lambda_r^p$ and $\zeta_r = \sum_{q:(p,q)\in E_C^-} \zeta_r^q$. Then, we can rewrite (LP1) as:

$$
\begin{aligned}
\text{maximize} & \sum_{p:(p,q)\in E_C^-} \sum_r \lambda_r^p v(X_r^p) + \sum_{q:(p,q)\in E_C^-} \sum_r \zeta_r^q v(Y_r^q) - \sum_{k\in K} \sum_{(p,q)\in E_C^-} |w_{pq}^k| y_{pq}^k \\
\text{subject to} & - \sum_{p:(p,q)\in E_C^-} \sum_r \lambda_r^p = -1 \\
& - \sum_{q:(p,q)\in E_C^-} \sum_r \zeta_r^q = -1 \\
& \sum_{p:(p,q)\in E_C^-} \sum_{r:X_{rk}^p=1} \lambda_r^p X_r^p + \sum_{q:(p,q)\in E_C^-} \sum_{r:Y_{rk}^q=1} \zeta_r^q Y_r^q \leq \\
& \qquad \sum_{(p,q)\in E_C^-} y_{pq}^k + |E_C^-| \quad \forall k \in K \\
& \lambda_r^p, \zeta_r^q, y_{pq}^k \geq 0 \; \forall r, k
\end{aligned}
$$

Notice that the constraint matrix of this linear program is again a network matrix, and thus totally unimodular. This is because each variable appears in at most two constraints with coefficient of opposite signs and absolute value 1, given the fact that X_{rk}^p and $X_{rk'}^p$ cannot be both 1. Therefore, there exists an integral solution in this program.

■

B All Negative & Sign-balanced but Sign-inconsistent Case

Figure 5 is a counter-example to the existence of a stable matching in the case with all negative total pairwise surplus values.

Figure 5: All negative case

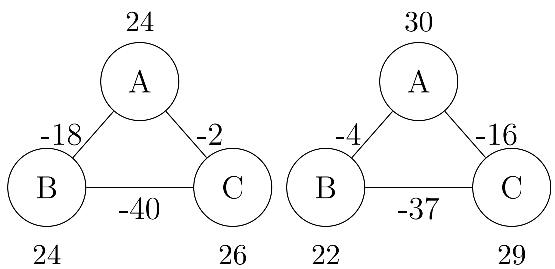

Figure 6 is a counter-example to the existence of a stable matching in the case with sign-balanced but sign-inconsistent graph.

Figure 6: Sign-balance but sign-inconsistent case